ANSWERS TO ANALYSIS III, FINAL EXAMINATION, 2008, B.MATH 2ND YEAR

1 (i) $U = GL(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R})$, $f : U \to M(n,\mathbb{R})$ is defined by

$$f(A) = A^{-1}$$

Claim: $Df(A)X = -A^{-1}XA^{-1}$

Note that $\lambda : M(n,\mathbb{R}) \to M(n,\mathbb{R})$ defined by $\lambda(X) = -A^{-1}XA^{-1}$ is a linear map and for C such that $||C||$ is small we have $(I + A^{-1}C)^{-1} = I - A^{-1}C + (A^{-1}C)^2 - \cdots$, hence we have

$$\lim_{C\to 0} \frac{||f(A+C) - f(A) - \lambda(C)||}{||C||}$$
$$= \lim_{C\to 0} \frac{||(A+C)^{-1} - A^{-1} + A^{-1}CA^{-1}||}{||C||}$$
$$= \lim_{C\to 0} \frac{||(I+A^{-1}C)^{-1}A^{-1} - A^{-1} + A^{-1}CA^{-1}||}{||C||}$$
$$= \lim_{C\to 0} \frac{||(I - A^{-1}C + (A^{-1}C)^2 - \cdots)A^{-1} - A^{-1} + A^{-1}CA^{-1}||}{||C||}$$
$$= \lim_{C\to 0} \frac{||(I - A^{-1}C + (A^{-1}C)^2 - \cdots - I + A^{-1}C)A^{-1}||}{||C||}$$
$$= \lim_{C\to 0} \frac{||((A^{-1}C)^2 - (A^{-1}C)^3 + \cdots)A^{-1}||}{||C||}$$
$$\le \lim_{C\to 0} \sum_{n=1}^{\infty} \frac{||A^{-1}||^{n+2}||C||^{n+1}}{||C||}$$
$$= 0$$

Thus $Df(A)X = -A^{-1}XA^{-1}$ for $X \in M(n,\mathbb{R})$

1 (ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \cos(xy) - y^2e^{-x^2}$$

Then f is a C^∞ function, $f(0,1) = 0$ and $D_2f(0,1) = -2 \neq 0$.
Therefore by implicit function theorem (smooth version) there is a C^∞ function $y = g(x)$ defined on some open neighbourhood A of 0 to a nighbourhood B of 1 such that for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$.
Thus $g(0) = 1, \cos(xg(x)) - g(x)^2e^{-x^2} = 0$.

2 (i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map and

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \text{ for all } x, y \in \mathbb{R}^n, t \in [0,1]$$

Let $x \in \mathbb{R}^n, x \neq 0$
$\Rightarrow f(tx) = tf(x) + (1-t)f(0)$
$\Rightarrow f(tx) - f(0) = t(f(x) - f(0))$

$\Rightarrow \lim_{t\to 0} \frac{||f(tx)-f(0)-f'(0)(tx)||}{||tx||} = 0$ since f is differentiable at 0
$\Rightarrow \lim_{t\to 0} \frac{||t(f(x)-f(0))-tf'(0)(x)||}{||tx||} = 0$
$\Rightarrow \lim_{t\to 0} \frac{||(f(x)-f(0))-f'(0)(x)||}{||x||} = 0$
$\Rightarrow f(x) - f(0) = f'(0)(x)$
Thus for all $x \in \mathbb{R}^n$ we have $f(x) = f'(0)x + f(0)$

2 (ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$
$D_1(f(0,0)) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$
$D_2(f(0,0)) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$
If f were differentiable at 0, then we shall have $f'(0) = \begin{bmatrix} 0 & 0 \end{bmatrix} = 0$, but
$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-f(0,0)-f'(0)(x,y)|}{|(x,y)|} = \lim_{(x,y)\to(0,0)} \frac{|x^2y|}{(x^2+y^2)^{\frac{3}{2}}} \neq 0$
(Considering the above limit along the direction $y = x$ we get the limit value as $\frac{1}{2^{\frac{3}{2}}}$)
Thus f is not differentiable at 0

3 (i) (a) $\omega(x_1,x_2,x_3) = \frac{x1dx_2\wedge dx_3+x_2dx_3\wedge dx_1+x_3dx_1\wedge dx_2}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}$

$$
\begin{aligned}
d\omega &= \frac{-1}{(x_1^2+x_2^2+x_3^2)^3} d(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}} \wedge (x_1dx_2\wedge dx_3 + x_2dx_3dx_1 + x_3dx_1\wedge dx_2) \\
&+ \frac{1}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}(dx_1\wedge dx_2\wedge dx_3 + dx_2\wedge dx_3\wedge dx_1 + dx_3\wedge dx_1\wedge dx_2) \\
&= \frac{-3}{2(x_1^2+x_2^2+x_3^2)^3}(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}(2x_1dx_1 + 2x_2dx_2 + 2x_3dx_3) \\
&\wedge (x_1dx_2\wedge dx_3 + x_2dx_3\wedge dx_1 + x_3dx_1\wedge dx_2) + \frac{3(dx_1\wedge dx_2\wedge dx_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} \\
&= \frac{-3(2x_1^2dx_1\wedge dx_2\wedge dx_3 + 2x_2^2dx_2\wedge dx_3\wedge dx_1 + 2x_3^2dx_3\wedge dx_1\wedge dx_2)}{2(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} \\
&+ \frac{3(dx_1\wedge dx_2\wedge dx_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} \\
&= \frac{-3(x_1^2+x_2^2+x_3^2)}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}dx_1\wedge dx_2\wedge dx_3 + \frac{3(dx_1\wedge dx_2\wedge dx_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} \\
&= 0
\end{aligned}
$$

3 (i) (b) σ is the singular 2-cube in $\mathbb{R}^3 \setminus \{0\}$ given by
$\sigma : [0,1]\times[0,1] \to \mathbb{R}^3 \setminus \{0\}$
$(s,t) \mapsto (\cos\pi(t-1/2)\cos 2\pi s, \cos\pi(t-1/2)\sin 2\pi s, \sin\pi(t-1/2))$
$\omega(x_1,x_2,x_3) = \frac{x1dx_2\wedge dx_3+x_2dx_3\wedge dx_1+x_3dx_1\wedge dx_2}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}$
$\frac{\partial}{\partial t}|_{(s,t)} \longrightarrow (-\pi\sin\pi(t-1/2)\cos 2\pi s\frac{\partial}{\partial x_1} - \pi\sin\pi(t-1/2)\sin 2\pi s\frac{\partial}{\partial x_2} + \pi\cos\pi(t-1/2)\frac{\partial}{\partial x_3})$
$\frac{\partial}{\partial s}|_{(s,t)} \longrightarrow (-2\pi\cos\pi(t-1/2)\sin 2\pi s\frac{\partial}{\partial x_1} + 2\pi\cos\pi(t-1/2)\cos 2\pi s\frac{\partial}{\partial x_2})$
$\sigma^* dx_1(\frac{\partial}{\partial t}) = dx_1(\sigma^*\frac{\partial}{\partial t}) = -\pi\sin\pi(t-1/2)\cos 2\pi s$
$\sigma^* dx_1(\frac{\partial}{\partial s}) = dx_1(\sigma^*\frac{\partial}{\partial s}) = -2\pi\cos\pi(t-1/2)\sin 2\pi s$

$$\sigma^*(dx_1) = -\pi \sin \pi(t-1/2) \cos 2\pi s \; dt - 2\pi \cos \pi(t-1/2) \sin 2\pi s \; ds$$
$$\sigma^*(dx_2) = -\pi \sin \pi(t-1/2) \sin 2\pi s \; dt + 2\pi \cos \pi(t-1/2) \cos 2\pi s \; ds$$
$$\sigma^*(dx_3) = \pi \cos \pi(t-1/2) \; dt$$

$$\begin{aligned}
\int_\sigma \omega &= \int_{[0,1]^2} \cos \pi(t-1/2) \cos 2\pi s(-2\pi^2 \cos^2 \pi(t-1/2) \cos 2\pi s) \; dt \wedge ds \\
&+ \cos \pi(t-1/2) \sin 2\pi s(-2\pi^2 \cos^2 \pi(t-1/2) \sin 2\pi s) \; dt \wedge ds \\
&+ \sin \pi(t-1/2)(-2\pi^2 \sin \pi(t-1/2) \cos \pi(t-1/2)) \; dt \wedge ds \\
&= \int_{[0,1]^2} -2\pi^2 \cos^3 \pi(t-1/2) \; dt \wedge ds - 2\pi^2 \sin^2 \pi(t-1/2) \cos \pi(t-1/2) \; dt \wedge ds \\
&= \int_{[0,1]^2} -2\pi^2 \cos \pi(t-1/2) \; dt \wedge ds \\
&= -2\pi^2 \int_0^1 \int_0^1 \cos \pi(t-1/2) \; dtds \\
&= -2\pi^2 \int_0^1 \frac{2}{\pi} ds \\
&= -4\pi
\end{aligned}$$

3 (ii) Given thet ω is the differential 2-form on $\mathbb{R}^2$ given by $\omega = xdx \wedge dy$ and v, w are the vector fields given by $v = y^2\partial_x + x^3\partial_y, w = 2x\partial_x - 3\partial_y$

$$\begin{aligned}
\omega(v,w) &= (xdx \wedge dy)(v,w) \\
&= 2\mathrm{Alt}(xdx \otimes dy)(v,w) \\
&= \frac{2}{2}\left[(xdx \otimes dy)(v,w) - (dy \otimes xdx)(v,w)\right] \\
&= xdx(v)dy(w) - dy(v)xdx(w) \\
&= xdx(y^2\partial_x + x^3\partial_y)dy(2x\partial_x - 3\partial_y) - dy(y^2\partial_x + x^3\partial_y)xdx(2x\partial_x - 3\partial_y) \\
&= xy^2(-3) - x^3(2x^2) \\
&= -2x^5 - 3xy^2
\end{aligned}$$

4 (i) Let $U \subset \mathbb{R}^n$ be an open set.
Let $\phi_i : \Omega^i(U) \to \Omega^{i+1}(U)$ be the operator defined by

$$\omega \mapsto \omega \wedge dx_1$$

Note that $\omega \in \Omega^i$ can be written uniquely as $\omega = \sum_{R \in J} f_R dx_R$ where $dx_R = dx_{r_1} \wedge dx_{r_2} \wedge ... \wedge dx_{r_i}$, $J = \{R = (r_1, r_2, ..., r_i) : n \geq r_1 > r_2 > \cdots > r_i \geq 1\}$ and $f_R : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Let $J' = \{R = (r_1, r_2, ..., r_i) \in J : r_i = 1\}$. For $R = (r_1, r_2, ..., r_i) \in J'$ let $R_1 = (r_1, r_2, ..., r_{i-1})$. Then $\omega = \sum_{R \in J'} f_R dx_{R_1} \wedge dx_1 + \sum_{R \in J \setminus J'} f_R dx_R$ where $\alpha := \sum_{R \in J \setminus J'} f_R dx_R \in \Omega^i, \beta := \sum_{R \in J'} f_R dx_{R_1} \in \Omega^{i-1}$ do not involve dx_1.

Clearly this expression is unique.
Assume that $\omega = \alpha + \beta \wedge dx_1 \in \ker \phi_i \Rightarrow \phi_i(\omega) = \alpha \wedge dx_1 + \beta dx_1 \wedge dx_1 = 0$
$\Rightarrow \alpha \wedge dx_1 = 0$
$\Rightarrow \alpha = 0$
$\Rightarrow \omega = \beta \wedge dx_1 = \phi_{i-1}(\beta)$
$\Rightarrow \omega \in \mathrm{Im}(\phi_{i-1})$
Conversely assume that $\omega \in \mathrm{Im}(\phi_{i-1})$
$\Rightarrow \omega = \phi_{i-1}(\beta) = \beta \wedge dx_1$ for some $\beta \in \Omega^{i-1}$
$\Rightarrow \phi_i(\omega) = \omega \wedge dx_1 = \beta \wedge dx_1 \wedge dx_1 = 0$
$\Rightarrow \omega \in \ker \phi_i$

4 (ii) $U = \mathbb{R}^2 \setminus \{0\}, P(x,y) = \frac{x - x^2y - y^3}{(x^2+y^2)^2}, Q(x,y) = \frac{x^3+y+xy^2}{(x^2+y^2)^2}$
Assume that there is a smooth function f on U satisfying the simultaneous first order PDE's $\frac{\partial f}{\partial x} = P(x,y), \frac{\partial f}{\partial y} = Q(x,y)$. Then $df = Pdx + Qdy$
Let C be the unit circle in $\mathbb{R}^2$. Then by Stokes theorem $\int_C df = 0$, but

$$
\begin{aligned}
\int_C df &= \int_C Pdx + Qdy \\
&= \int_C \frac{x - x^2y - y^3}{(x^2+y^2)^2} dx + \int_C \frac{x^3 + y + xy^2}{(x^2+y^2)^2} dy \\
&= \int_0^{2\pi} \frac{\cos\theta - \cos^2\theta \sin\theta - \sin^3\theta}{(\sin^2\theta + \cos^2\theta)^2} (-\sin\theta) d\theta + \int_0^{2\pi} \frac{\cos^3\theta + \sin\theta + \cos\theta \sin^2\theta}{(\sin^2\theta + \cos^2\theta)^2} (\cos\theta) d\theta \\
&= \int_0^{2\pi} d\theta \\
&= 2\pi
\end{aligned}
$$

Thus there no smooth function f on U satisfying the simultaneous first order PDE's $\frac{\partial f}{\partial x} = P(x,y), \frac{\partial f}{\partial y} = Q(x,y)$.

5 (i) Let $\sigma_i : [0,1] \to \mathbb{R}^2, i = 1,2,3,4$ be the singular 1-cubes
$\sigma_1(t) = (2t^2 - 2t + 1, 2t - 1); \sigma_2(t) = \left(\sqrt{2}\cos\frac{\pi(2t+1)}{4}, \sqrt{2}\sin\frac{\pi(2t+1)}{4}\right);$
$\sigma_3(t) = (-1, 1 - 3t); \sigma_4(t) = (2t - 1, t - 2)$
Let $\sigma = \sum_{i=1}^4 \sigma_i, \ \omega = xdy - ydx$

$\int_\sigma \omega = \sum_{i=1}^4 \int_\sigma \omega = \sum_{i=1}^4 \int_0^1 \sigma_i^*(\omega)$

$$\begin{aligned}
\int_0^1 \sigma_1^*(\omega) = \int_0^1 \sigma_1^*(xdy - ydx) &= \int_0^1 \sigma_1^*(x)\sigma_1^*(dy) - \sigma_1^*(y)\sigma_1^*(dx) \\
&= \int_0^1 (2t^2 - 2t + 1)d(\sigma_1^*(y)) - (2t-1)d(\sigma_1^*(x)) \\
&= \int_0^1 (2t^2 - 2t + 1)d(2t-1) - (2t-1)d(2t^2 - 2t + 1) \\
&= \int_0^1 (2t^2 - 2t + 1)2dt - (2t-1)(4tdt - 2dt) \\
&= \int_0^1 (2t^2 - 2t + 1)2dt - (2t-1)(4tdt - 2dt) \\
&= \int_0^1 4(t - t^2)dt \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sigma_2^*(\omega) &= \int_0^1 \sigma_2^*(xdy - ydx) = \int_0^1 \sigma_2^*(x)\sigma_2^*(dy) - \sigma_2^*(y)\sigma_2^*(dx) \\
&= \int_0^1 (\sqrt{2}\cos\frac{\pi(2t+1)}{4})d(\sigma_1^*(y)) - (\sqrt{2}\sin\frac{\pi(2t+1)}{4})d(\sigma_1^*(x)) \\
&= \int_0^1 \pi(\cos\frac{\pi(2t+1)}{4})\cos\frac{\pi(2t+1)}{4}dt + \pi(\sin\frac{\pi(2t+1)}{4})\sin\frac{\pi(2t+1)}{4}dt \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sigma_3^*(\omega) = \int_0^1 \sigma_3^*(xdy - ydx) &= \int_0^1 \sigma_3^*(x)\sigma_3^*(dy) - \sigma_3^*(y)\sigma_3^*(dx) \\
&= \int_0^1 (-1)d(1-3t) - (1-3t)d(-1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sigma_4^*(\omega) = \int_0^1 \sigma_4^*(xdy - ydx) &= \int_0^1 \sigma_4^*(x)d(\sigma_4^*(y)) - \sigma_4^*(y)d(\sigma_4^*(x)) \\
&= \int_0^1 (2t-1)d(t-2) - (t-2)d(2t-1) \\
&= 3
\end{aligned}$$

Therefore $\int_\sigma \omega = \sum_{i=1}^4 \int_\sigma \omega = \sum_{i=1}^4 \int_0^1 \sigma_i^*(\omega) = \frac{2}{3} + \pi + 3 + 3 = \frac{20+3\pi}{3}$

5 (ii) Let M be the region enclosed by σ. Then $\partial M = \sigma$

Let $\alpha, \beta : M \to \mathbb{R}$ be defined by

$$\alpha(x,y) = -y, \beta(x,y) = x$$

Then α, β are differentiable. By Green's theorem for manifolds with corner, we have

$$\int\int_M dxdy = \int\int_M \frac{1}{2}(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y})dxdy = \frac{1}{2}\int_\sigma (\alpha dx + \beta dy)$$

$$= \frac{1}{2}\int_\sigma (xdy - ydx) = \frac{1}{2}\int_\sigma \omega = \frac{20 + 3\pi}{6} \text{ by part (i)}$$